Practical tests with irregular and regular finite spectra of a proposed statistical measure for quantum chaos

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Using the first $N=668$ measured eigenfrequencies of a two-dimensional (2D) microwave cavity, we test experimentally the properties of a quantity $W(x)$ proposed by Aurich, Bolte, and Steiner [Phys. Rev. Lett. **73**, 1356 (1994)] as a statistical measure for quantum chaos in spectra. Our data confirm that the distribution of $W(x)$ for the spectrum of the classically irregular cavity has a statistically significant Gaussian form. We also calculate spectra of classically regular 2D cavities (rectangular and square) up to comparable values of N and calculate their $W(x)$ distributions. Finding that their distributions, too, are close to Gaussian form, we conclude that one should not expect to be able to use the distribution of $W(x)$ as an effective experimental tool for deciding whether a given *finite quantum spectrum* corresponds to a classically irregular (chaotic) or regular (integrable) system. [S1063-651X(96)51607-7]

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Investigation of quantum systems whose classical counterparts are chaotic is the subject of quantum chaos, a dynamically growing field [1,2]. So far, because theoretical work in quantum chaos has greatly exceeded experimental work, many theoretical predictions lack empirical confirmation. The experimental work we report in this Rapid Communication on the distribution of the quantity $W(x)$ proposed in [3] as a statistical measure for chaos in spectra is directed toward closing this gap.

In contrast to a bounded classical Hamiltonian system (compact phase space) that is chaotic, i.e., its evolution is exponentially sensitive to the initial conditions, the corresponding bounded quantum system has a discrete eigenenergy spectrum and evolves quasiperiodically. Nevertheless, there has been a vigorous search for signatures of chaos in quantum systems. The fingerprints of classical chaos were discovered in the distribution of eigenenergies of the corresponding quantum system. It was found that the eigenenergy distribution for even a low-dimensional ($\geqslant 2$) classically chaotic system can be described by random matrix theory (RMT) [4]. Studies in RMT have emphasized the Gaussian orthogonal (GOE), Gaussian unitary (GUE), and Gaussian symplectic (GSE) ensembles. The GOE and GSE [GUE] pertain(s) to physics that is [is not] invariant under time reversal. All three are characterized by level repulsion: at small spacings linear for GOE, quadratic for GUE, and degree four for GSE.

Predictions of RMT have been confirmed in two-dimensional (2D) quantum billiards [1,5,6]. Of great experimental importance for bounded systems is that the 2D Schrödinger equation is equivalent to the 2D Helmholtz equation for electromagnetic waves, $(\nabla^2+k^2)\Psi=0$, where $\mathbf{k}$ is a wave vector. This allows analogs of 2D bounded quantum systems to be studied experimentally with 2D microwave [7–9] or acoustic [10] cavities. In particular, GOE and GUE statistics, respectively, have been confirmed for microwave cavities that are [7–9] (are not [11,12]) time-reversal invariant. GOE statistics were also found for 3D irregularly shaped microwave cavities [13], even though the physics of the 3D Schrödinger (scalar wave) equation and its boundary conditions are not mathematically equivalent to those for the 3D Helmholtz equation for (vector) electromagnetic waves. GSE statistics have been found for numerically computed eigenvalues of the lattice Dirac operator in quantum chromodynamics [14].

Quantum spectra of classically integrable systems are believed to have the statistics of Poissonian random proceses; these reflect a tendency toward level clustering, although a careful statistical analysis showed important deviations from the ''pure'' Poisson case [15].

In addition to the results described above, there are classically chaotic systems having quantal counterparts whose spectral statistics are not those of RMT. One well known example is geodesic flows on hyperbolic surfaces, which exhibit so-called arithmetical chaos [16,17]. This example does not have the quantal spectral statistics of the (nonarithmetical) strongly chaotic systems described above; rather, it has near-Poisson quantal spectral statistics, such as is ''expected'' for classically regular systems. Therefore, there is a clear need for new, quantitative statistical measure(s) of quantum chaos in spectra. Such measure(s) should discriminate neatly between regular and irregular spectra that, we emphasize, will be *finite* in all *practical* cases, i.e., obtained experimentally or numerically.

A recent paper [3] presented a quantity to ''measure quantum chaos in spectra,'' viz., $W(x)=\mathcal{N}_{fl}(x)/\sqrt{\Delta_\infty(x)}$; $\mathcal{N}_{fl}(x)$ is the fluctuating part of the spectral staircase function $\mathcal{N}(x)=\bar{\mathcal{N}}(x)+\mathcal{N}_{fl}(x)$, $\bar{\mathcal{N}}(x)$ is the smooth Weyl term describing the ''mean behavior'' and $\Delta_\infty(x)=\lim_{L\to\infty}\Delta_3(L,x)$, where $\Delta_3(L,x)$ is the spectral rigidity. For the case considered in this paper, $x=\sqrt{E}$, where E is the energy above the ground state.

The results of Ref. [3] for the asympototic ($x\to\infty$) distribution of $W(x)$ may be summarized as follows: (i) for scaling, strongly chaotic, bound classical systems, including those exhibiting arithmetical chaos, the distribution should be a Gaussian; (ii) for classically integrable systems, it should be non-Gaussian.

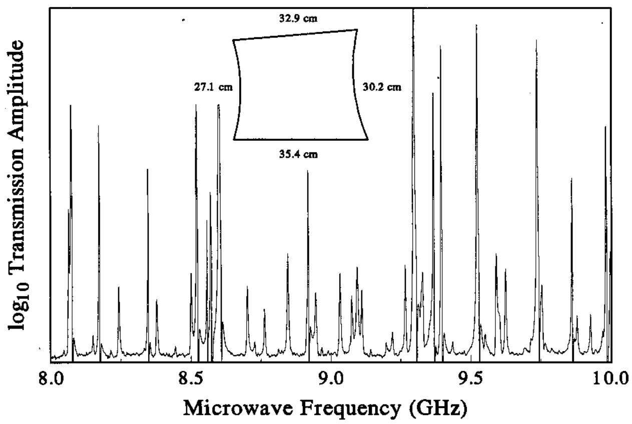

FIG. 1. A portion of the measured frequency spectrum for a 2D microwave cavity (shape shown in the inset) that simulates a classically chaotic 2D quantum billiard. Note: the tops of two resonance peaks were ''clipped'' to make room for the inset.

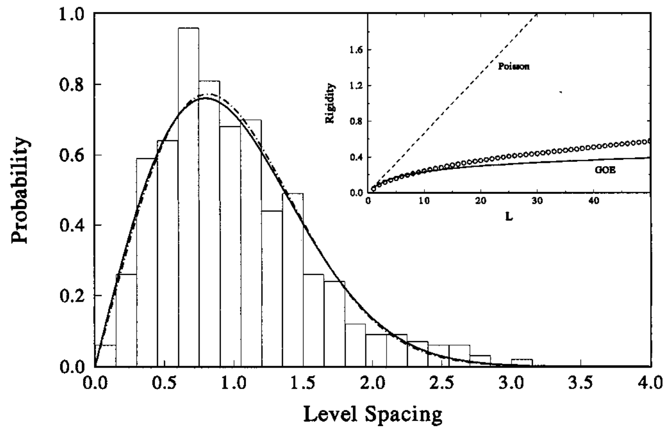

FIG. 2. The nearest-neighbor level-spacing distribution for the cavity shown in Fig. 1 (inset). The full line shows the GOE prediction; the dot-dashed line shows a Brody distribution (see the text) fitted to the experimental distribution, giving $\beta=1.05(9)$. The inset shows the experimental spectral rigidity $\Delta_3(L)$ compared to GOE (full line) and Poisson (dashed line) predictions.

An obvious disadvantage of using this spectral measure is that one is supposed to construct $W(x)$ asymptotically for $x\rightarrow\infty$ and then study its distribution for both x and $L\rightarrow\infty$. Because any experimental or numerical spectrum is, perforce, finite, the practical utility of $W(x)$ will hinge on its *nonasymptotic* behavior.

Using *finite spectra* obtained numerically, the authors of Ref. [3] presented supporting evidence for their conjectures for $W(x)$, see above, for three different chaotic systems: (a) geodesic flow on a nonarithmethic compact hyperbolic surface of genus two, using the 4500th to 6000th eigenvalues; (b) a billiard on the hyperbolic plane that shows arithmetical chaos, using the first 1040 eigenvalues; (c) a truncated hyperbola billiard on the Euclidean plane, using the first 1850 eigenvalues. In all three cases the Gaussian form of the distributions $W(x)$ was found to be statistically significant even though the results were obtained with finite x values, i.e., were nonasymptotic.

In experiments simulating 2D quantum billiards with 2D microwave cavities used at room temperature, one typically obtains well-resolved spectra up to 600–700 eigenfrequencies above the ground state. [Because the leading term in $\bar{\mathcal{N}}(\nu)\propto\nu^2$ and because each level has a finite frequency width $\Delta\nu\sim\nu/Q$ given by the cavity quality factor Q, there must exist a ν^* above which one can no longer cleanly resolve levels.]

To our knowledge, this Rapid Communication reports the first experimental test of the usefulness of the quantity $W(x)$ as a statistical measure for quantum chaos in a classically chaotic system. Our experiment used a microwave cavity (inset in Fig. 1) that was built to simulate a classically chaotic quantum billiard. The cavity had an area of 0.0886(2) m^2 and a perimeter of 1.256(2) m. Its height $d=6.4$ mm was chosen to ensure two-dimensionality up to $\nu_{max}=c/2d\simeq23.4$ GHz. Constructed of polished brass, the cavity had a quality factor $Q\simeq2\times10^3$. The cavity had four sidewalls, two convex with average radius around 1 m and two straight but nonparallel; this geometry ensured that its classical periodic orbits were unstable (hard chaos) and isolated.

Microwave power was coupled into and out of the cavity via coaxial cables terminated by small loops that were inserted into the cavity through small holes (3.6 mm diameter) located on one of the side walls at the cavity midplane. Both the size and the insertion depth of each loop were fixed empirically as a compromise between coupling strong enough to excite/detect even weak resonances and coupling weak enough to avoid strong perturbation of the cavity.

Over the frequency range 0.5–18 GHz we recorded transmission spectra for the cavity [19] and stored them in a computer. We ensured that we did not miss weakly excited and/or detected resonances by recording spectra for several different positions of the coupling antennas [13]. We were able to resolve cleanly 668 eigenfrequencies from the ground state at 0.762 GHz up to a maximum of 14.998 GHz. Figure 1 shows the 8–10 GHz portion of the spectrum, with the logarithmic vertical scale bringing out weaker resonances having amplitudes 2–3 orders of magnitude below the strongest one.

We analyzed the transmision spectra in two different ways: (1) After unfolding the spectrum [13], we calculated the nearest-neighbor-spacing distribution [1,20] and spectral rigidity $\Delta_3(L)$ [1] and compared them with GOE predictions. For this type of analysis, the formula $E_n=k_n^2=(2\pi/c)^2\nu_n^2$ relates energies E_n to measured resonance frequencies ν_n. (2) We calculated the distribution of the function $W(x)$ introduced in [3]. Here $x_n=k_n$.

Figure 2 and its inset show, respectively, the nearest-neighbor-spacing distribution and spectral rigidity $\Delta_3(L)$ for our 2D cavity. For the former we performed a least-squares fit to the empirical Brody distribution [21], $P_\beta(s)=as^\beta\exp(-bs^{\beta+1})$, where s is a level spacing normalized to the local average level spacing, β is a (level repulsion) parameter, and $a=(\beta+1)b$, $b=\{\Gamma[(\beta+2)/(\beta+1)]\}^{\beta+1}$, Γ being the gamma function. The fit yields $\beta=1.05\pm0.09$; see the dot-dash line in Fig. 2. Because $\beta=0$ [$\beta=1$] corresponds to the Poisson [GOE] level statistics, our results, as expected, agree with the GOE case (solid line in Fig. 2).

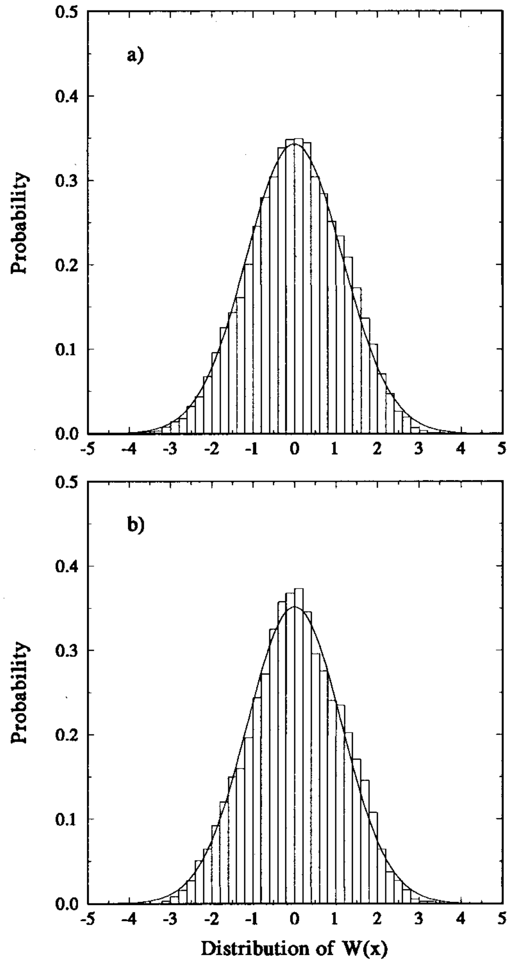

FIG. 3. Experimentally obtained distributions of $W(x)$ $[=\mathcal{W}(w)]$ for the lowest 668 levels of the cavity shown in Fig. 1 (inset) compared with fitted Gaussian distributions: panel (a), $\mathcal{W}(w)$ obtained for $L=100$; panel (b), $\mathcal{W}(w)$ obtained for $L=200$.

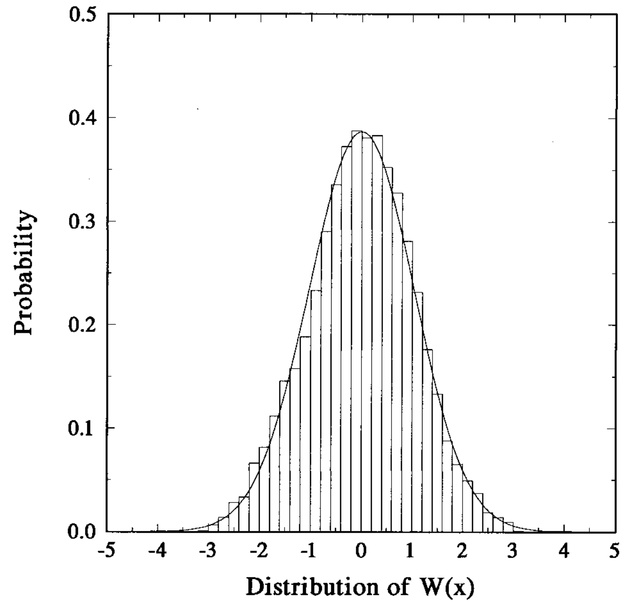

FIG. 4. Theoretically calculated distribution of $W(x)$ $[=\mathcal{W}(w)]$ for the lowest 657 levels of a rectangular billiard (see the text) compared with a fitted Gaussian. The calculations were done for $L=200$.

The inset of Fig. 2 compares the spectral rigidity $\Delta_3(L)$ for our cavity data with the theoretical prediction for GOE (full line) and Poisson (dashed line) statistics. The small departure of the data from the GOE prediction confirms that our experimental system is classically strongly chaotic.

Figure 3 shows the distribution $\mathcal{W}(w)$ of the quantity $W(x)$ [3].

Because of the finite width of each resonance and, at higher frequencies, the requirement that one maintain two-dimensionality, it is an inescapable experimental fact that one must deal with finite x values; one cannot pass experimentally to the asymptotic regime of x and $L \rightarrow \infty$. (This is also the case for numerically computed spectra; cf. [3].) Therefore, we approximated $\Delta_\infty(x)$ by determining $\Delta_3(L,x)$ for several experimentally accessible L values. Figure 3(a) [Fig. 3(b)] shows $\mathcal{W}(w)$ obtained for $L=100$ [$L=200$]. Fitting a Gaussian function $G(w)=\sqrt{A/\pi}\exp(-Aw^2)+B$, where A and B are fitting parameters, to $\mathcal{W}(w)$ gave $A=0.369(5)$ [0.388(7)], and $B=0.0000(6)$ [0.0000(8)] for $L=100$ [$L=200$]. The mean variance $\mathcal{V}=\Sigma_{i=1}^{i=n}[\mathcal{W}(w_i)-G(w_i)]^2/n$, where $\mathcal{W}(w_i)$ is the value of $\mathcal{W}$ in the ith bin, and n is the total number of bins where $\mathcal{W}(w_i)$ is nonzero, measures the departure of the experimental $\mathcal{W}(w)$ from the fitted $G(w)$. We found $\mathcal{V}$ to be 0.97×10^{-4} [2.03×10^{-4}] for the results presented in Fig. 3(a) [Fig. 3(b)].

Qualitatively, visual inspection of Figs. 3(a) and 3(b) and quantitatively, the small values of $\mathcal{V}$ show that our $\mathcal{W}(w)$ data are approximated well by Gaussian functions. However, note that the variance of $\mathcal{W}(w)$, $\mathrm{var}[\mathcal{W}]=\int_{-\infty}^{+\infty}dw\,w^2\mathcal{W}(w)$, which is 1.28 for Fig. 3(a) and 1.21 for Fig. 3(b), has not yet reached the asymptotic value of 1 predicted by theory [3]. We also used the spectral entropy $\mathcal{E}[\mathcal{W}]=-\int_{-\infty}^{+\infty}dw\,\mathcal{W}(w)\ln\mathcal{W}(w)$ as a quantitative measure for spectral randomness [3]. Using our data, we obtained 1.54 for $L=100$ and 1.50 for $L=200$.

Given the strictly limited number of energy levels available experimentally, a crucial question is whether the method presented in [3] for measuring chaos in spectra can distinguish clearly between classically integrable and classically chaotic quantum systems. To check this, we calculated $\mathcal{W}(w)$ for two integrable systems: a rectangular cavity (RC) and a square cavity (SC) that simulated rectangular and square billiards, respectively. We chose the areas of RC, 25×35 cm^2, and of SC, 29.77×29.77 cm^2, to be close to the area 886(2) cm^2 of our experimental chaotic cavity (CC). Over the frequency range 0–15 GHz, CC (RC) [SC] has 668 (657) [667] eigenfrequencies, which guarantees that one will be comparing statistics computed for nearly equal numbers of levels.

Figure 4 shows $\mathcal{W}(w)$ for RC. Fitting a Gaussian curve $G(w)$ gives $A=0.470(6)$ and $B=0.0000(7)$. Note that we calculated $\mathcal{W}(w)$ for RC for the same range of x and L used for our experimental CC data in Fig. 3(b), in particular, for $L=200$. Calculations for RC give a mean variance $\mathcal{V}=1.45\times10^{-4}$, $\mathrm{var}[\mathcal{W}]=1.06$, and $\mathcal{E}[\mathcal{W}]=1.44$.

Qualitatively, visual inspection of Fig. 4 shows that $G(w)$ reasonably approximates the ''regular'' RC data. Quantitatively, the small value of $\mathcal{V}$ confirms this. We obtained similar results (not shown here) for SC.

We remark that the widths of the experimental $\mathcal{W}(w)$ distributions shown in Figs. 3(a) and 3(b) for CC are somewhat larger than that in Fig. 4 for the integrable RC. Similarly, the spectral entropy for CC is several percent larger than that for RC.

We conclude that if one is given a finite spectrum for an unknown system, it will be very difficult to use only the similarity, or lack thereof, of its $\mathcal{W}(w)$ to a Gaussian distribution to decide whether the underlying classical dynamics is integrable or nonintegrable.

We believe that this somewhat surprising result, obtained for classically integrable systems where one would expect to get a much narrower distribution, is connected with the properties of $W(x)$. Compared to the classically chaotic case (level repulsion), level-spacing fluctuations are bigger for classically integrable systems (level clustering); additionally, the spectral rigidity $\Delta_3(L)$ for the regular case is found to saturate at large L at the value (much) below $L/15$ [15,18]. For example, for the first N levels of RC one can use the results of Ref. [15] (its Eq. 3 with $\epsilon_{cr}=0.5$) to estimate that $\Delta_{3sat}\simeq 0.06\sqrt{N}$; for $N=657$, this yields the estimate $\Delta_{3sat}\simeq 1.5$. Our result obtained by averaging of $\Delta_3(L,x)$ over x for $L=200$ for the RC spectrum used for Fig. 4 is the same, viz., 1.51.

In summary, for a classically chaotic 2D quantum billiard simulated by a 2D microwave cavity, we have evaluated distributions of a function $W(x)$ introduced as a spectral measure for chaos in spectra. We found that the distribution of $W(x)$ is approximated well by a Gaussian distribution. However, theoretically calculated distributions of $W(x)$ for classically integrable quantum systems, such as those simulated by rectangular and square cavities, are also close to Gaussian. Given the inevitable, practical restriction of having only a finite number of levels in our experimental spectrum, we found that the shape of the distribution of $W(x)$ is not sensitive enough to be used as a practical diagnostic tool for distinguishing between classically chaotic and classically integrable quantum systems only on the basis of the first $N\simeq 650$ levels. Whether this will continue to be the case for much larger values of N can only be addressed when one is able to obtain (experimentally or numerically) well-resolved spectra with many more levels.

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